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REDUNDANCY OPTIMIZATION FOR SERIES K-OUT-OF-N SYSTEMS

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ABSTRACT

This analysis considers the optimum allocation of redundancy in a system of serially-connected subsystems in which each subsystem is of the k-out-of-n type. The two problems treated are (a) maximization of system reliability subject to multiple cost constraints and (b) minimization of some function of multiple costs while maintaining at least a minimum acceptable level of reliability.

Five techniques are presented for solving one or both of these problems. Since the choice of solution technique is determined by such things as computer program availability, degree of accuracy needed, and extent of desired optimum surface mapping, the relative merits of these techniques are discussed.

The series-parallel redundancy optimization problem is a special case of the problem treated here. For this case, approximate solutions in closed form are presented. These solutions are compared with solutions produced by a method of exact optimization.

INTRODUCTION

There is a basic conflict in utilizing redundancy in a system. The addition of redundant components increases the costs - measured in

weight, volume, money, etc. - at the same time that it increases reliability. Redundancy optimization is an effort to minimize this conflict.

The simplest redundancy arrangement is one for which n components are connected in parallel and for which the required function is performed as long as at least one of the components functions properly. If all components are identical and equally susceptible to failure, the arrangement is called parallel redundancy. When several such parallel subsystems are serially connected, the familiar series-parallel system results as seen in figure 1. It is for this particular system that most of the redundancy optimization procedures have been developed.

A more general redundancy arrangement is one which requires at least k of n components to function properly for subsystem success. The serial connection of several such subsystems results in the series k -out-of- n system as seen in figure 2.

The problem of optimally allocating redundancy in the series k -out-of- n system is treated herein. Study of this problem was motivated because (1) k -out-of- n subsystems are important by themselves and (2) k -out-of- n subsystems include as special cases the parallel and series subsystems.

Finally, closed-form approximate equations are given for optimum redundancy allocation for the series-parallel system.

REDUNDANCY ALLOCATION PROBLEMS

Problem I: Maximizing reliability subject to multiple cost constraints

The function to be maximized is the system reliability R given by

$$R = \prod_{i=1}^m r_i \quad (1)$$

where r_i is the reliability of the i^{th} subsystem and there are m such subsystems connected in series. Associated with each component of subsystem i there are s different cost factors, none of which can exceed the allowable resource $C_{\cdot j \text{ max}}$. Hence, the constraint function is given by

$$\sum_{i=1}^m c_{ij} n_i \leq C_{\cdot j \text{ max}} \quad (j=1, s; n_i \text{ integer}) \quad (2)$$

where c_{ij} is the j^{th} type of cost of a single component of the i^{th} subsystem, n_i is the number of components in the i^{th} subsystem, and where the \cdot subscript denotes summation over $i=1, m$.

Problem II: Minimizing some function of multiple costs while maintaining a minimum acceptable system reliability

The function to be minimized is some function of the s cost factors

$$C_{\cdot j} = \sum_{i=1}^m c_{ij} n_i \quad (j=1, s; n_i \text{ integer}) \quad (3)$$

subject to the constraint that

$$\prod_{i=1}^m r_i \geq R_{\min} \quad (4)$$

SUBSYSTEM RELIABILITY EQUATIONS

For simple parallel subsystems the subsystem reliability is

$$r_i = 1 - q_i^{n_i} \quad (5)$$

where q_i is the failure probability of the i^{th} component.

For k-out-of-n subsystems, the subsystem reliability is given by the binomial summation

$$r_i = \sum_{x=k_i}^{n_i} \binom{n_i}{x} p_i^x (1 - p_i)^{n_i-x} \quad (k_i, n_i \text{ integer}) \quad (6)$$

where $p_i = 1 - q_i$ is the success probability of the i^{th} component.

In principle, the optimum allocation of redundancy can be achieved by examining all possible combinations of components and choosing the ones which satisfy the requirements of reliability and/or costs. This, however, involves an inordinate expenditure of effort. Several methods have been used to obtain solutions for the series-parallel system without searching all combinations. Five of these methods have been adapted to the problem of series k-out-of-n systems and these methods are briefly discussed. A more detailed account of these techniques is contained in Bien (1970).

METHODS OF SOLUTION

The first method is due to Kettelle (1962) and was modified by Proschan and Bray (1965) to include multiple cost factors. Their work, of course, is for the series-parallel system model. Kettelle's modified method is used to generate the complete family of solutions over a range of reliability and costs but, since it is a dynamic programming technique, it is plagued by the problem of dimensionality. The second method uses Lagrange multipliers as applied to nondifferentiable functions (Everett, 1963) to generate the "best" solutions with much less effort than Kettelle's method. The third method involves selecting consecutively for redundancy the subsystem which contributes the greatest reliability

per unit of weighted costs (Barlow and Proschan, 1965). By generating solutions for a grid of weighting factors, this method gives the same solutions as the Lagrange multiplier method. These last two methods are subject to some restrictions on the subsystem reliability equation.

Two other methods are available for treating Problem I, the maximization of reliability subject to several cost constraints. Ghare and Taylor (1969) use a multi-dimensional knapsack formulation of the series-parallel system optimization problem. Using a branch-and-bound procedure, they obtain the exact solution to Problem I. It is shown in Bien (1970) that their procedure is also valid for the series k-out-of-n system. The final method of solving this problem is due to Mizukami (1968). For maximizing concave functions, his method is referred to as the method of concave and integer programming. The reliability function which is to be maximized is made approximately piecewise linear in his paper. Methods of linear programming can thus be used to obtain approximate solutions to this problem.

EXAMPLE PROBLEM

Shown in figure 3 is an example of a series k-out-of-n system consisting of four subsystems. The component success probability, monetary cost, and weight are shown in the figure. The values of k_i are 3, 2, 1, and 4 as denoted by the lack of shading. This example problem was solved by the first three methods and the results are shown in figure 4.

The complete set of configurations having system cost between 115 and 130 while system reliability is between 0.90 and 0.95 literally fills the triangular area below and to the right of the solid line in figure 4.

Only the optimum or undominated solutions are noted in the figure. That is, any other configuration is either more costly for the same reliability or less reliable for the same costs.

COMPARISON OF METHODS

The method of Kettelle (1962) provides the complete family of undominated allocations to be redundancy optimization problems. It is a dynamic programming procedure and, hence, becomes quite unwieldy for large systems subject to many constraints. It is readily adapted to computer analysis, however, and Proschan and Bray (1965) discuss a computer program capable of handling a maximum of three constraints, a maximum of sixty-four subsystems, a maximum of ten components in each subsystem, and a maximum of 1024 entries in the dominating set at any combination of subsystems. According to them, the only method for determining whether it is practical to solve a given problem is to attempt to find the solution. Kettelle (1962) and Proschan and Bray (1965) both introduced an assumption in producing their dominating sets, resulting in an error in reliability no more than $(1 - R)^2$. That unnecessary assumption was eliminated from the development presented in Bien (1970), the procedure producing exact solutions.

A partial list of undominated solutions comprising the convex-hull, or "best", optimum allocations is determined by the method of Lagrange multipliers due to Everett (1963). A trial-and-error procedure is required in the selection of the Lagrange multipliers; the multipliers yielding the optimal solutions are not known beforehand but are produced in the course of the solution. This technique is most useful in determining the single best allocation satisfying the constraints rather

than in generating the optimum solutions over a range of the constraints. The complexity increases substantially with the number of constraints, an obvious disadvantage for the many-constraint problem.

The same convex-hull points are produced more readily by the method of balancing sensitivities. Because the procedure begins with subsystems whose reliability is at the lower end of a range of interest, the procedure is particularly suited to the case for which a set of solutions is desired satisfying a range of constraints. The convex hull solutions are produced with much less effort than by the method of Kettelle (1962). The trial and error involved in selecting the appropriate weighting factors is minimal compared to the trial and error involved in the method of Lagrange multipliers. Since all combinations of weighting factors must be investigated in order to assure that none of the possible solutions is missed, there is an obvious problem of dimensionality for problems involving many constraints.

The last two procedures produce only the convex hull points and, as such, they may miss the optimal allocations of interest. It is suggested that one of these last two methods be used to produce an optimal allocation whose reliability or cost vector is at the lower end of the range of interest. This allocation could then be used as a starting point for the dynamic programming procedure and the successively larger redundancy allocations could be produced with much less effort than if just the dynamic programming procedure alone were used.

The method of Ghare and Taylor (1969) produces exact solutions to Problem I, the problem of maximizing reliability subject to several cost constraints. They have written a computer program solving problems

of up to 100 subsystems, up to 15 constraints, and up to 500 redundant components for the series parallel system. Their program requires only 5500 words of memory space on the IBM 360/50 system.

The method of integer concave programming (Mizukami, 1968) produces approximate solutions to Problem I. The advantage of this technique is that standard linear programming methods can be used. For integer solutions, he suggests Gomory's (1958) integer linear programming technique. The approximations introduced by this method can be made arbitrarily good but at a cost in complexity.

THE SERIES-PARALLEL SYSTEM

The method of Lagrange multipliers was used to obtain closed-form solutions for series-parallel redundancy optimization when only a single type of cost is of interest. Solutions thus obtained are approximate in that the resultant numbers of components are not necessarily integral. However, such non-integer solutions may be good enough for most purposes; if not, they form a convenient basis for an integer programming formulation.

Maximizing Reliability for Fixed Cost

For the series-parallel system, the function to be maximized is

$$R = \prod_{i=1}^m (1 - q_i^{x_i}) \quad (7)$$

subject to

$$\sum_{i=1}^m c_i x_i \leq C \quad (8)$$

where x_i is the continuous variable for number of components in

subsystem i . The resultant solution by Lagrange multipliers, which is good for reasonably reliable systems, is

$$x_i \approx \frac{1}{-\ln q_i} \left\{ -\ln \left(\frac{c_i}{-\ln q_i} \right) + \frac{C + \sum_{i=1}^m \left[\left(\frac{c_i}{-\ln q_i} \right) \ln \left(\frac{c_i}{-\ln q_i} \right) \right]}{\sum_{i=1}^m \left(\frac{c_i}{-\ln q_i} \right)} \right\} \quad (9)$$

Minimizing Cost for Fixed Reliability

The function to be minimized is

$$C = \sum_{i=1}^m c_i x_i \quad (10)$$

subject to

$$\prod_{i=1}^m (1 - q_i^{x_i}) \geq R \quad (11)$$

The resultant equation for x_i in terms of R is

$$x_i \approx \frac{1}{-\ln q_i} \left\{ -\ln (1 - R) - \ln \left[\frac{(c_i / -\ln q_i)}{\sum_{i=1}^m (c_i / -\ln q_i)} \right] \right\} \quad (12)$$

Equations (9) and (12) allow ready determination of x_i in terms of all known quantities. An example problem was solved to check the usefulness of these equations. The system is shown in figure 5. In figure 6 the optimum solutions produced by the methods of Kettelle (1962) and Everett (1963) are shown along with the curve of system reliability obtained using x_i as a function of C from equation (9).

In the search for optimal solutions, a reasonable question concerns the maximum reliability achievable for a fixed cost or the minimum cost achievable for a fixed reliability. These equations allow rapid determination of those optimum conditions theoretically achievable with non-integer numbers of components. In this way a systems designer can determine whether a given design (arrangement of components) is near optimum and hence whether much could be gained by searching for the unique optimum design.

SUMMARY AND DISCUSSION

The optimization of redundancy in the series k-out-of-n system can be accomplished by appropriate use of some of the techniques used to solve similar problems for series-parallel systems. The two types of optimization problems considered herein are:

- (1) Maximizing system reliability subject to multiple cost constraints, and

- (2) Minimizing some function of the multiple cost factors subject to maintaining at least a minimum acceptable level of system reliability

Five techniques are discussed for solving one or both of the above problems without resorting to the inordinate expenditure of effort required in examining all possible combinations of components. These five methods are:

- (1) Kettelle's (1962) dynamic programming procedure as modified by Proschan and Bray (1965) to include multiple constraints,

- (2) Everett's (1963) generalized Lagrange multiplier technique,

(3) The method of selecting consecutively for redundancy those subsystems which increase the reliability per unit of weighted costs by the greatest amount; e.g., Barlow and Proschan (1965).

(4) Ghare and Taylor's (1969) formulation of the problem as a multi-dimensional knapsack problem using a branch-and-bound procedure, and

(5) Mizukami's (1968) combined concave and integer programming procedure.

These five methods are compared and an example problem is solved by the first three methods.

Closed-form equations, derived by the use of Lagrange multipliers, are presented for the series-parallel system optimization problems of

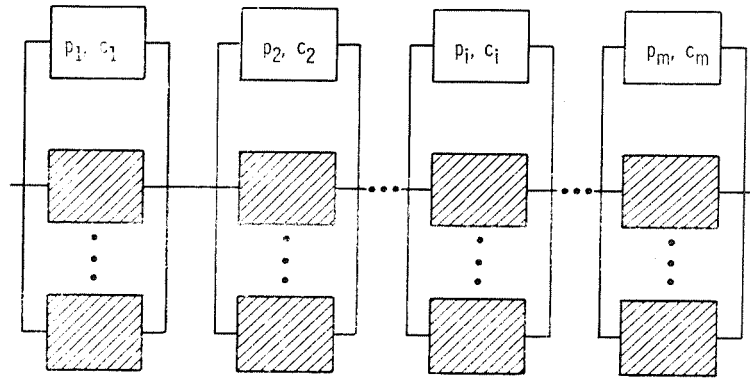
(1) Maximizing system reliability while maintaining a fixed single cost constraint, and

(2) Minimizing a single cost while maintaining a fixed system reliability.

The usefulness of the equations is shown by means of an example problem.

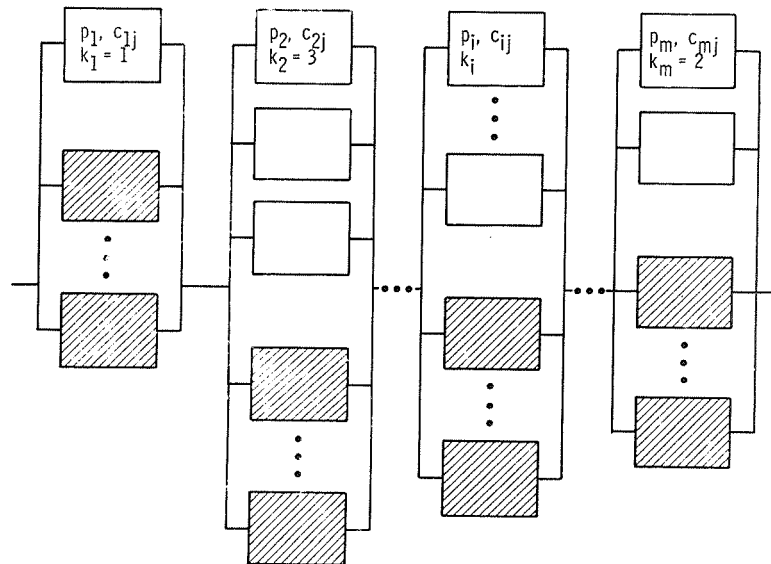
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Shading indicates redundant component

Figure 1. - Typical series parallel system.



Shading indicates redundant component

Figure 2 - Typical series k -out-of- n system.

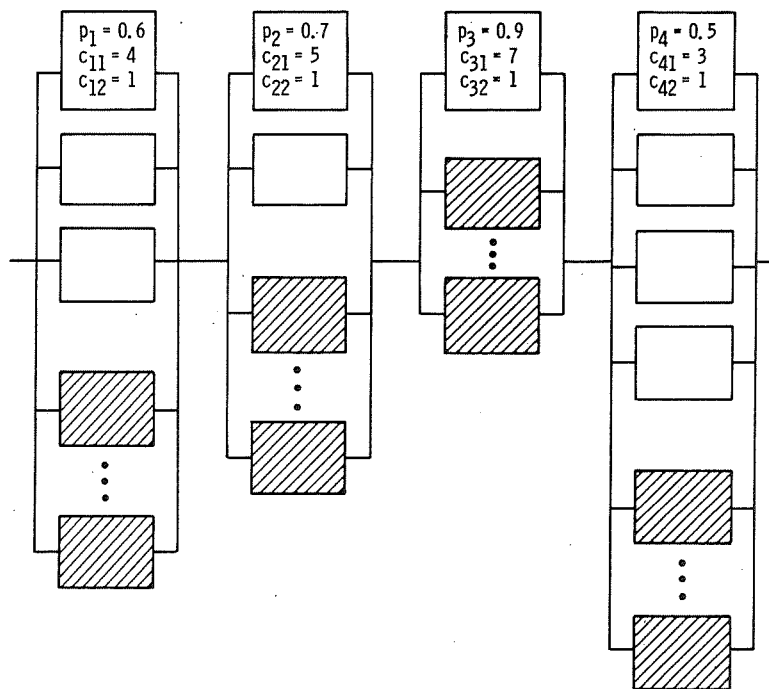


Figure 3. - System of 4 k-out-of-n subsystems connected in series; example problem.

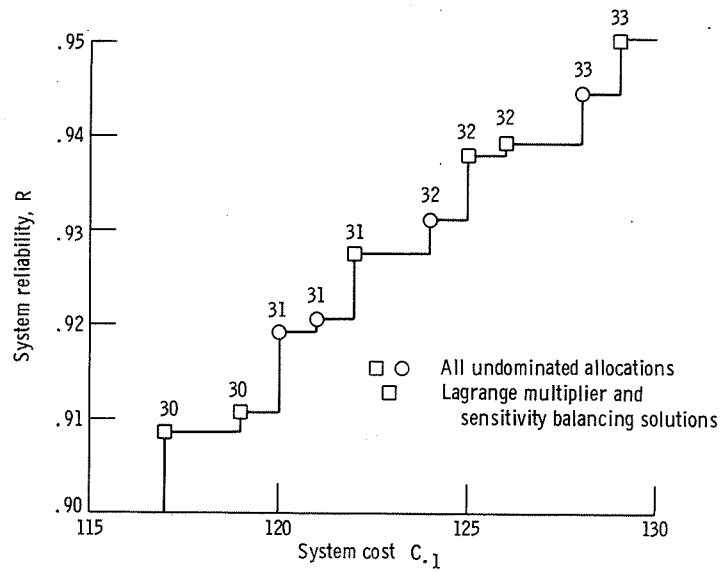
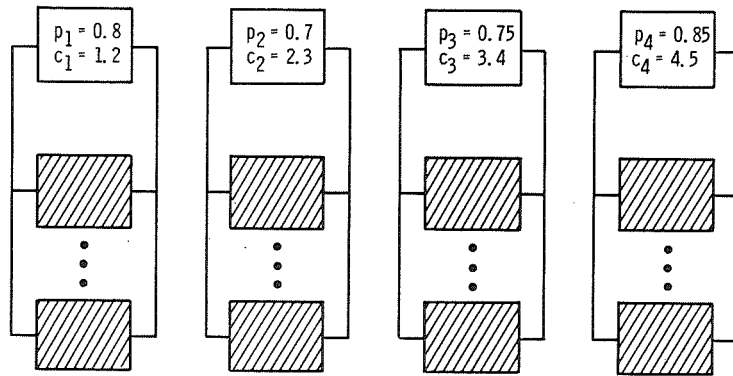


Figure 4. - Optimum solutions to example problem showing R as a function of C_1 and C_2 as the parameter.



Shading indicates redundant component

Figure 5. - System of 4 parallel subsystems connected in series, example problem.

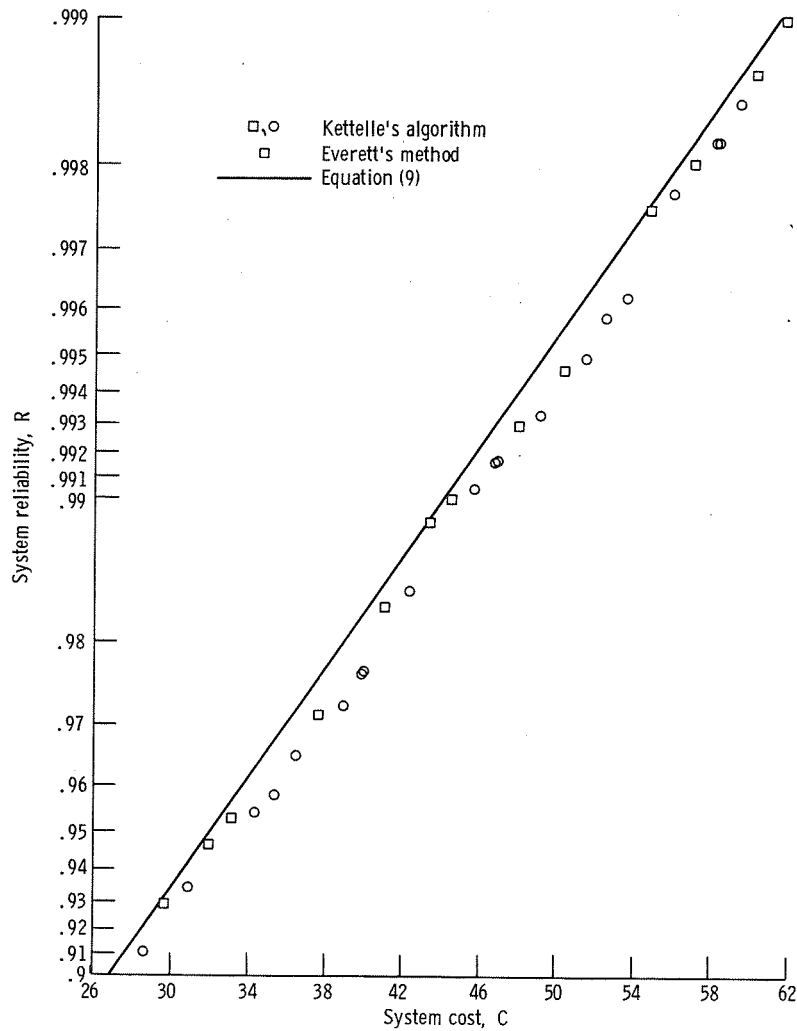


Figure 6. - Comparison of exact and approximate solutions, example problem.